

1

On Sense and Reference

Gottlob Frege

Equality* gives rise to challenging questions which are not altogether easy to answer. Is it a relation? A relation between objects, or between names or signs of objects? In my *Begriffsschrift*¹ I assumed the latter. The reasons which seem to favour this are the following: $a = a$ and $a = b$ are obviously statements of differing cognitive value; $a = a$ holds *a priori* and, according to Kant, is to be labelled analytic, while statements of the form $a = b$ often contain very valuable extensions of our knowledge and cannot always be established *a priori*. The discovery that the rising sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even to-day the identification of a small planet or a comet is not always a matter of course. Now if we were to regard equality as a relation between that which the names '*a*' and '*b*' designate, it would seem that $a = b$ could not differ from $a = a$ (i.e. provided $a = b$ is true). A relation would thereby be expressed of a thing to itself, and indeed one in which each thing stands to itself but to no other thing. What is intended to be said by $a = b$ seems to be that the signs or names '*a*' and '*b*' designate the same thing, so that those signs themselves would be under discussion; a relation between them would be asserted. But this relation would hold between the names or signs only in so far as they named or designated something. It would be mediated by the connexion of each of the two signs with the same designated thing. But this is arbitrary. Nobody can be forbidden to use any arbitrarily producible event or object as a sign for something. In that case the sentence $a = b$ would no longer refer to the subject matter, but only to its mode of designation; we would express no proper knowledge by its

* I use this word in the sense of identity, and understand ' $a = b$ ' to have the sense of '*a* is the same as *b*' or '*a* and *b* coincide.'

means. But in many cases this is just what we want to do. If the sign 'a' is distinguished from the sign 'b' only as object (here, by means of its shape), not as sign (i.e. not by the manner in which it designates something), the cognitive value of $a = a$ becomes essentially equal to that of $a = b$, provided $a = b$ is true. A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated. Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c . So we have different designations for the same point, and these names ('point of intersection of a and b ,' 'point of intersection of b and c ') likewise indicate the mode of presentation; and hence the statement contains actual knowledge.

It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. In our example, accordingly, the reference of the expressions 'the point of intersection of a and b ' and 'the point of intersection of b and c ' would be the same, but not their senses. The reference of 'evening star' would be the same as that of 'morning star,' but not the sense.

It is clear from the context that by 'sign' and 'name' I have here understood any designation representing a proper name, which thus has as its reference a definite object (this word taken in the widest range), but not a concept or a relation, which shall be discussed further in another article.² The designation of a single object can also consist of several words or other signs. For brevity, let every such designation be called a proper name.

The sense of a proper name is grasped by everybody who is sufficiently familiar with the language or totality of designations to which it belongs;* but this serves to illuminate only a single aspect of the reference, supposing it to have one. Comprehensive knowledge of the reference would require us to be able to say immediately whether any given sense belongs to it. To such knowledge we never attain.

* In the case of an actual proper name such as 'Aristotle' opinions as to the sense may differ. It might, for instance, be taken to be the following: the pupil of Plato and teacher of Alexander the Great. Anybody who does this will attach another sense to the sentence 'Aristotle was born in Stagira' than will a man who takes as the sense of the name: the teacher of Alexander the Great who was born in Stagira. So long as the reference remains the same, such variations of sense may be tolerated, although they are to be avoided in the theoretical structure of a demonstrative science and ought not to occur in a perfect language.

The regular connexion between a sign, its sense, and its reference is of such a kind that to the sign there corresponds a definite sense and to that in turn a definite reference, while to a given reference (an object) there does not belong only a single sign. The same sense has different expressions in different languages or even in the same language. To be sure, exceptions to this regular behaviour occur. To every expression belonging to a complete totality of signs, there should certainly correspond a definite sense; but natural languages often do not satisfy this condition, and one must be content if the same word has the same sense in the same context. It may perhaps be granted that every grammatically well-formed expression representing a proper name always has a sense. But this is not to say that to the sense there also corresponds a reference. The words 'the celestial body most distant from the Earth' have a sense, but it is very doubtful if they also have a reference. The expression 'the least rapidly convergent series' has a sense but demonstrably has no reference, since for every given convergent series, another convergent, but less rapidly convergent, series can be found. In grasping a sense, one is not certainly assured of a reference.

If words are used in the ordinary way, what one intends to speak of is their reference. It can also happen, however, that one wishes to talk about the words themselves or their sense. This happens, for instance, when the words of another are quoted. One's own words then first designate words of the other speaker, and only the latter have their usual reference. We then have signs of signs. In writing, the words are in this case enclosed in quotation marks. Accordingly, a word standing between quotation marks must not be taken as having its ordinary reference.

In order to speak of the sense of an expression 'A' one may simply use the phrase 'the sense of the expression "A"'. In reported speech one talks about the sense, e.g., of another person's remarks. It is quite clear that in this way of speaking words do not have their customary reference but designate what is usually their sense. In order to have a short expression, we will say: In reported speech, words are used *indirectly* or have their *indirect* reference. We distinguish accordingly the *customary* from the *indirect* reference of a word; and its *customary* sense from its *indirect* sense. The indirect reference of a word is accordingly its customary sense. Such exceptions must always be borne in mind if the mode of connexion between sign, sense, and reference in particular cases is to be correctly understood.

The reference and sense of a sign are to be distinguished from the associated idea. If the reference of a sign is an object perceivable by the

senses, my idea of it is an internal image,* arising from memories of sense impressions which I have had and acts, both internal and external, which I have performed. Such an idea is often saturated with feeling; the clarity of its separate parts varies and oscillates. The same sense is not always connected, even in the same man, with the same idea. The idea is subjective: one man's idea is not that of another. There result, as a matter of course, a variety of differences in the ideas associated with the same sense. A painter, a horseman, and a zoologist will probably connect different ideas with the name 'Bucephalus.' This constitutes an essential distinction between the idea and the sign's sense, which may be the common property of many and therefore is not a part or a mode of the individual mind. For one can hardly deny that mankind has a common store of thoughts which is transmitted from one generation to another.†

In the light of this, one need have no scruples in speaking simply of *the* sense, whereas in the case of an idea one must, strictly speaking, add to whom it belongs and at what time. It might perhaps be said: Just as one man connects this idea, and another that idea, with the same word, so also one man can associate this sense and another that sense. But there still remains a difference in the mode of connexion. They are not prevented from grasping the same sense; but they cannot have the same idea. *Si duo idem faciunt, non est idem.* If two persons picture the same thing, each still has his own idea. It is indeed sometimes possible to establish differences in the ideas, or even in the sensations, of different men; but an exact comparison is not possible, because we cannot have both ideas together in the same consciousness.

The reference of a proper name is the object itself which we designate by its means; the idea, which we have in that case, is wholly subjective; in between lies the sense, which is indeed no longer subjective like the idea, but is yet not the object itself. The following analogy will perhaps clarify these relationships. Somebody observes the Moon through a telescope. I compare the Moon itself to the reference; it is the object of the observation, mediated by the real image projected by the object glass in the interior of the telescope, and by the retinal image of the observer.

* We can include with ideas the direct experiences in which sense-impressions and acts themselves take the place of the traces which they have left in the mind. The distinction is unimportant for our purpose, especially since memories of sense-impressions and acts always go along with such impressions and acts themselves to complete the perceptual image. One may on the other hand understand direct experience as including any object, in so far as it is sensibly perceptible or spatial.

† Hence it is inadvisable to use the word 'idea' to designate something so basically different.

The former I compare to the sense, the latter is like the idea or experience. The optical image in the telescope is indeed one-sided and dependent upon the standpoint of observation; but it is still objective, inasmuch as it can be used by several observers. At any rate it could be arranged for several to use it simultaneously. But each one would have his own retinal image. On account of the diverse shapes of the observers' eyes, even a geometrical congruence could hardly be achieved, and an actual coincidence would be out of the question. This analogy might be developed still further, by assuming A's retinal image made visible to B; or A might also see his own retinal image in a mirror. In this way we might perhaps show how an idea can itself be taken as an object, but as such is not for the observer what it directly is for the person having the idea. But to pursue this would take us too far afield.

We can now recognize three levels of difference between words, expressions, or whole sentences. The difference may concern at most the ideas, or the sense but not the reference, or, finally, the reference as well. With respect to the first level, it is to be noted that, on account of the uncertain connexion of ideas with words, a difference may hold for one person, which another does not find. The difference between a translation and the original text should properly not overstep the first level. To the possible differences here belong also the colouring and shading which poetic eloquence seeks to give to the sense. Such colouring and shading are not objective, and must be evoked by each hearer or reader according to the hints of the poet or the speaker. Without some affinity in human ideas art would certainly be impossible; but it can never be exactly determined how far the intentions of the poet are realized.

In what follows there will be no further discussion of ideas and experiences; they have been mentioned here only to ensure that the idea aroused in the hearer by a word shall not be confused with its sense or its reference.

To make short and exact expressions possible, let the following phraseology be established:

A proper name (word, sign, sign combination, expression) *expresses* its sense, *stands for* or *designates* its reference. By means of a sign we express its sense and designate its reference.

Idealists or sceptics will perhaps long since have objected: 'You talk, without further ado, of the Moon as an object; but how do you know that the name 'the Moon' has any reference? How do you know that anything whatsoever has a reference?' I reply that when we say 'the Moon,' we do

not intend to speak of our idea of the Moon, nor are we satisfied with the sense alone, but we presuppose a reference. To assume that in the sentence 'The Moon is smaller than the Earth' the idea of the Moon is in question, would be flatly to misunderstand the sense. If this is what the speaker wanted, he would use the phrase 'my idea of the Moon.' Now we can of course be mistaken in the presupposition, and such mistakes have indeed occurred. But the question whether the presupposition is perhaps always mistaken need not be answered here; in order to justify mention of the reference of a sign it is enough, at first, to point out our intention in speaking or thinking. (We must then add the reservation: provided such reference exists.)

So far we have considered the sense and reference only of such expressions, words, or signs as we have called proper names. We now inquire concerning the sense and reference for an entire declarative sentence. Such a sentence contains a thought.* Is this thought, now, to be regarded as its sense or its reference? Let us assume for the time being that the sentence has reference. If we now replace one word of the sentence by another having the same reference, but a different sense, this can have no bearing upon the reference of the sentence. Yet we can see that in such a case the thought changes; since, e.g., the thought in the sentence 'The morning star is a body illuminated by the Sun' differs from that in the sentence 'The evening star is a body illuminated by the Sun.' Anybody who did not know that the evening star is the morning star might hold the one thought to be true, the other false. The thought, accordingly, cannot be the reference of the sentence, but must rather be considered as the sense. What is the position now with regard to the reference? Have we a right even to inquire about it? Is it possible that a sentence as a whole has only a sense, but no reference? At any rate, one might expect that such sentences occur, just as there are parts of sentences having sense but no reference. And sentences which contain proper names without reference will be of this kind. The sentence 'Odysseus was set ashore at Ithaca while sound asleep' obviously has a sense. But since it is doubtful whether the name 'Odysseus,' occurring therein, has reference, it is also doubtful whether the whole sentence has one. Yet it is certain, nevertheless, that anyone who seriously took the sentence to be true or false would ascribe to the name 'Odysseus' a reference, not merely a sense; for it is of the reference of the name that the predicate is affirmed or denied. Whoever does not admit the name has reference can

* By a thought I understand not the subjective performance of thinking but its objective content, which is capable of being the common property of several thinkers.

neither apply nor withhold the predicate. But in that case it would be superfluous to advance to the reference of the name; one could be satisfied with the sense, if one wanted to go no further than the thought. If it were a question only of the sense of the sentence, the thought, it would be unnecessary to bother with the reference of a part of the sentence; only the sense, not the reference, of the part is relevant to the sense of the whole sentence. The thought remains the same whether 'Odysseus' has reference or not. The fact that we concern ourselves at all about the reference of a part of the sentence indicates that we generally recognize and expect a reference for the sentence itself. The thought loses value for us as soon as we recognize that the reference of one of its parts is missing. We are therefore justified in not being satisfied with the sense of a sentence, and in inquiring also as to its reference. But now why do we want every proper name to have not only a sense, but also a reference? Why is the thought not enough for us? Because, and to the extent that, we are concerned with its truth value. This is not always the case. In hearing an epic poem, for instance, apart from the euphony of the language we are interested only in the sense of the sentences and the images and feelings thereby aroused. The question of truth would cause us to abandon aesthetic delight for an attitude of scientific investigation. Hence it is a matter of no concern to us whether the name 'Odysseus,' for instance, has reference, so long as we accept the poem as a work of art.* It is the striving for truth that drives us always to advance from the sense to the reference.

We have seen that the reference of a sentence may always be sought, whenever the reference of its components is involved; and that this is the case when and only when we are inquiring after the truth value.

We are therefore driven into accepting the *truth value* of a sentence as constituting its reference. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True, the other the False. Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. These two objects are recognized, if only implicitly, by everybody who judges something to be true – and so even by a sceptic. The designation of the truth values as objects may appear to be an arbitrary fancy or perhaps a mere play upon words, from which no profound consequences could be drawn. What I mean by an object can

* It would be desirable to have a special term for signs having only sense. If we name them, say, representations, the words of the actors on the stage would be representations; indeed the actor himself would be a representation

be more exactly discussed only in connexion with concept and relation. I will reserve this for another article.³ But so much should already be clear, that in every judgment,* no matter how trivial, the step from the level of thoughts to the level of reference (the objective) has already been taken.

One might be tempted to regard the relation of the thought to the True not as that of sense to reference, but rather as that of subject to predicate. One can, indeed, say: 'The thought, that 5 is a prime number, is true.' But closer examination shows that nothing more has been said than in the simple sentence '5 is a prime number.' The truth claim arises in each case from the form of the declarative sentence, and when the latter lacks its usual force, e.g., in the mouth of an actor upon the stage, even the sentence 'The thought that 5 is a prime number is true' contains only a thought, and indeed the same thought as the simple '5 is a prime number.' It follows that the relation of the thought to the True may not be compared with that of subject to predicate.

Subject and predicate (understood in the logical sense) are indeed elements of thought; they stand on the same level for knowledge. By combining subject and predicate, one reaches only a thought, never passes from sense to reference, never from a thought to its truth value. One moves at the same level but never advances from one level to the next. A truth value cannot be a part of a thought, any more than, say, the Sun can, for it is not a sense but an object.

If our supposition that the reference of a sentence is its truth value is correct, the latter must remain unchanged when a part of the sentence is replaced by an expression having the same reference. And this is in fact the case. Leibniz gives the definition: '*Eadem sunt, quae sibi mutuo substitui possunt, salva veritate.*' What else but the truth value could be found, that belongs quite generally to every sentence if the reference of its components is relevant, and remains unchanged by substitutions of the kind in question?

If now the truth value of a sentence is its reference, then on the one hand all true sentences have the same reference and so, on the other hand, do all false sentences. From this we see that in the reference of the sentence all that is specific is obliterated. We can never be concerned only with the reference of a sentence; but again the mere thought alone yields no knowledge, but only the thought together with its reference, i.e. its truth value. Judgments can be regarded as advances from a thought to a truth value. Naturally this cannot be a definition. Judgment is something

* A judgment, for me, is not the mere comprehension of a thought, but the admission of its truth.

quite peculiar and incomparable. One might also say that judgments are distinctions of parts within truth values. Such distinction occurs by a return to the thought. To every sense belonging to a truth value there would correspond its own manner of analysis. However, I have here used the word 'part' in a special sense. I have in fact transferred the relation between the parts and the whole of the sentence to its reference, by calling the reference of a word part of the reference of the sentence, if the word itself is a part of the sentence. This way of speaking can certainly be attacked, because the whole reference and one part of it do not suffice to determine the remainder, and because the word 'part' is already used in another sense of bodies. A special term would need to be invented.

The supposition that the truth value of a sentence is its reference shall now be put to further test. We have found that the truth value of a sentence remains unchanged when an expression is replaced by another having the same reference: but we have not yet considered the case in which the expression to be replaced is itself a sentence. Now if our view is correct, the truth value of a sentence containing another as part must remain unchanged when the part is replaced by another sentence having the same truth value. Exceptions are to be expected when the whole sentence or its part is direct or indirect quotation; for in such cases, as we have seen, the words do not have their customary reference. In direct quotation, a sentence designates another sentence, and in indirect quotation a thought.

We are thus led to consider subordinate sentences or clauses. These occur as parts of a sentence complex, which is, from the logical standpoint, likewise a sentence – a main sentence. But here we meet the question whether it is also true of the subordinate sentence that its reference is a truth value. Of indirect quotation we already know the opposite. Grammarians view subordinate clauses as representatives of parts of sentences and divide them accordingly into noun clauses, adjective clauses, adverbial clauses. This might generate the supposition that the reference of a subordinate clause was not a truth value but rather of the same kind as the reference of a noun or adjective or adverb – in short, of a part of a sentence, whose sense was not a thought but only a part of a thought. Only a more thorough investigation can clarify the issue. In so doing, we shall not follow the grammatical categories strictly, but rather group together what is logically of the same kind. Let us first search for cases in which the sense of the subordinate clause, as we have just supposed, is not an independent thought.

The case of an abstract⁴ noun clause, introduced by 'that,' includes the case of indirect quotation, in which we have seen the words to have their

indirect reference coinciding with what is customarily their sense. In this case, then, the subordinate clause has for its reference a thought, not a truth value; as sense not a thought, but the sense of the words 'the thought, that...,' which is only a part of the thought in the entire complex sentence. This happens after 'say,' 'hear,' 'be of the opinion,' 'be convinced,' 'conclude,' and similar words.* There is a different, and indeed somewhat complicated, situation after words like 'perceive,' 'know,' 'fancy,' which are to be considered later.

That in the cases of the first kind the reference of the subordinate clause is in fact the thought can also be recognized by seeing that it is indifferent to the truth of the whole whether the subordinate clause is true or false. Let us compare, for instance, the two sentences 'Copernicus believed that the planetary orbits are circles' and 'Copernicus believed that the apparent motion of the sun is produced by the real motion of the Earth.' One subordinate clause can be substituted for the other without harm to the truth. The main clause and the subordinate clause together have as their sense only a single thought, and the truth of the whole includes neither the truth nor the untruth of the subordinate clause. In such cases it is not permissible to replace one expression in the subordinate clause by another having the same customary reference, but only by one having the same indirect reference, i.e. the same customary sense. If somebody were to conclude: The reference of a sentence is not its truth value, for in that case it could always be replaced by another sentence of the same truth value; he would prove too much; one might just as well claim that the reference of 'morning star' is not Venus, since one may not always say 'Venus' in place of 'morning star.' One has the right to conclude only that the reference of a sentence is not *always* its truth value, and that 'morning star' does not always stand for the planet Venus, viz. when the word has its indirect reference. An exception of such a kind occurs in the subordinate clause just considered which has a thought as its reference.

If one says 'It seems that...,' one means 'It seems to me that...,' or 'I think that...,' We therefore have the same case again. The situation is similar in the case of expressions such as 'to be pleased,' 'to regret,' 'to approve,' 'to blame,' 'to hope,' 'to fear.' If, toward the end of the battle of Waterloo,⁵ Wellington was glad that the Prussians were coming, the basis for his joy was a conviction. Had he been deceived, he would have been no less pleased so long as his illusion lasted; and before he became so convinced he could not have been pleased that the Prussians

* In 'A lied in saying he had seen B,' the subordinate clause designates a thought which is said (1) to have been asserted by A (2) while A was convinced of its falsity.

were coming – even though in fact they might have been already approaching.

Just as a conviction or a belief is the ground of a feeling, it can, as in inference, also be the ground of a conviction. In the sentence: ‘Columbus inferred from the roundness of the Earth that he could reach India by travelling towards the west,’ we have as the reference of the parts two thoughts, that the Earth is round, and that Columbus by travelling to the west could reach India. All that is relevant here is that Columbus was convinced of both, and that the one conviction was a ground for the other. Whether the Earth is really round and Columbus could really reach India by travelling west, as he thought, is immaterial to the truth of our sentence; but it is not immaterial whether we replace ‘the Earth’ by ‘the planet which is accompanied by a moon whose diameter is greater than the fourth part of its own.’ Here also we have the indirect reference of the words.

Adverbial final clauses beginning ‘in order that’ also belong here; for obviously the purpose is a thought; therefore: indirect reference for the words, subjunctive mood.

A subordinate clause with ‘that’ after ‘command,’ ‘ask,’ ‘forbid,’ would appear in direct speech as an imperative. Such a clause has no reference but only a sense. A command, a request, are indeed not thoughts, yet they stand on the same level as thoughts. Hence in subordinate clauses depending upon ‘command,’ ‘ask,’ etc., words have their indirect reference. The reference of such a clause is therefore not a truth value but a command, a request, and so forth.

The case is similar for the dependent question in phrases such as ‘doubt whether,’ ‘not to know what.’ It is easy to see that here also the words are to be taken to have their indirect reference. Dependent clauses expressing questions and beginning, with ‘who,’ ‘what,’ ‘where,’ ‘when,’ ‘how,’ ‘by what means,’ etc., seem at times to approximate very closely to adverbial clauses in which words have their customary references. These cases are distinguished linguistically [in German] by the mood of the verb. With the subjunctive, we have a dependent question and indirect reference of the words, so that a proper name cannot in general be replaced by another name of the same object.

In the cases so far considered the words of the subordinate clauses had their indirect reference, and this made it clear that the reference of the subordinate clause itself was indirect, i.e. not a truth value but a thought, a command, a request, a question. The subordinate clause could be regarded as a noun, indeed one could say: as a proper name of that thought, that command, etc., which it represented in the context of the sentence structure.

We now come to other subordinate clauses, in which the words do have their customary reference without however a thought occurring as sense and a truth value as reference. How this is possible is best made clear by examples.

Whoever discovered the elliptic form of the planetary orbits died in misery.

If the sense of the subordinate clause were here a thought, it would have to be possible to express it also in a separate sentence. But this does not work, because the grammatical subject 'whoever' has no independent sense and only mediates the relation with the consequent clause 'died in misery.' For this reason the sense of the subordinate clause is not a complete thought, and its reference is Kepler, not a truth value. One might object that the sense of the whole does contain a thought as part, viz. that there was somebody who first discovered the elliptic form of the planetary orbits; for whoever takes the whole to be true cannot deny this part. This is undoubtedly so; but only because otherwise the dependent clause 'whoever discovered the elliptic form of the planetary orbits' would have no reference. If anything is asserted there is always an obvious presupposition that the simple or compound proper names used have reference. If one therefore asserts 'Kepler died in misery,' there is a presupposition that the name 'Kepler' designates something; but it does not follow that the sense of the sentence 'Kepler died in misery' contains the thought that the name 'Kepler' designates something. If this were the case the negation would have to run not

Kepler did not die in misery

but

Kepler did not die in misery, or the name 'Kepler' has no reference.

That the name 'Kepler' designates something is just as much a presupposition for the assertion

Kepler died in misery

as for the contrary assertion. Now languages have the fault of containing expressions which fail to designate an object (although their grammatical form seems to qualify them for that purpose) because the truth of

some sentence is a prerequisite. Thus it depends on the truth of the sentence:

There was someone who discovered the elliptic form of the planetary orbits

whether the subordinate clause

Whoever discovered the elliptic form of the planetary orbits

really designates an object or only seems to do so while having in fact no reference. And thus it may appear as if our subordinate clause contained as a part of its sense the thought that there was somebody who discovered the elliptic form of the planetary orbits. If this were right the negation would run:

Either whoever discovered the elliptic form of the planetary orbits did not die in misery or there was nobody who discovered the elliptic form of the planetary orbits.

This arises from an imperfection of language, from which even the symbolic language of mathematical analysis is not altogether free; even there combinations of symbols can occur that seem to stand for something but have (at least so far) no reference, e.g. divergent infinite series. This can be avoided, e.g., by means of the special stipulation that divergent infinite series shall stand for the number 0. A logically perfect language (*Begriffsschrift*) should satisfy the conditions, that every expression grammatically well constructed as a proper name out of signs already introduced shall in fact designate an object, and that no new sign shall be introduced as a proper name without being secured a reference. The logic books contain warnings against logical mistakes arising from the ambiguity of expressions. I regard as no less pertinent a warning against apparent proper names having no reference. The history of mathematics supplies errors which have arisen in this way. This lends itself to demagogic abuse as easily as ambiguity – perhaps more easily. ‘The will of the people’ can serve as an example; for it is easy to establish that there is at any rate no generally accepted reference for this expression. It is therefore by no means unimportant to eliminate the source of these mistakes, at least in science, once and for all. Then such objections as the one discussed above would become impossible, because it could never depend upon the truth of a thought whether a proper name had a reference.

With the consideration of these noun clauses may be coupled that of types of adjective and adverbial clauses which are logically in close relation to them.

Adjective clauses also serve to construct compound proper names, though, unlike noun clauses, they are not sufficient by themselves for this purpose. These adjective clauses are to be regarded as equivalent to adjectives. Instead of 'the square root of 4 which is smaller than 0,' one can also say 'the negative square root of 4.' We have here the case of a compound proper name constructed from the expression for a concept with the help of the singular definite article. This is at any rate permissible if the concept applies to one and only one single object.*

Expressions for concepts can be so constructed that marks of a concept are given by adjective clauses as, in our example, by the clause 'which is smaller than 0.' It is evident that such an adjective clause cannot have a thought as sense or a truth value as reference, any more than the noun clause could. Its sense, which can also be expressed in many cases by a single adjective, is only a part of a thought. Here, as in the case of the noun clause, there is no independent subject and therefore no possibility of reproducing the sense of the subordinate clause in an independent sentence.

Places, instants, stretches of time, are, logically considered, objects; hence the linguistic designation of a definite place, a definite instant, or a stretch of time is to be regarded as a proper name. Now adverbial clauses of place and time can be used for the construction of such a proper name in a manner similar to that which we have seen in the case of noun and adjective clauses. In the same way, expressions for concepts bringing in places, etc., can be constructed. It is to be noted here also that the sense of these subordinate clauses cannot be reproduced in an independent sentence, since an essential component, viz. the determination of place or time, is missing and is only indicated by a relative pronoun or a conjunction.†

In conditional clauses, also, there may usually be recognized to occur an indefinite indicator, having a similar correlate in the dependent clause. (We have already seen this occur in noun, adjective, and adverbial

* In accordance with what was said above, an expression of the kind in question must actually always be assured of reference, by means of a special stipulation, e.g. by the convention that 0 shall count as its reference, when the concept applies to no object or to more than one.

† In the case of these sentences, various interpretations are easily possible. The sense of the sentence, 'After Schleswig-Holstein was separated from Denmark, Prussia and Austria quarrelled' can also be rendered in the form 'After the separation of Schleswig-Holstein

clauses.) In so far as each indicator refers to the other, both clauses together form a connected whole, which as a rule expresses only a single thought. In the sentence

If a number is less than 1 and greater than 0, its square is less than 1 and greater than 0

the component in question is 'a number' in the conditional clause and 'its' in the dependent clause. It is by means of this very indefiniteness that the sense acquires the generality expected of a law. It is this which is responsible for the fact that the antecedent clause alone has no complete thought as its sense and in combination with the consequent clause expresses one and only one thought, whose parts are no longer thoughts. It is, in general, incorrect to say that in the hypothetical judgment two judgments are put in reciprocal relationship. If this or something similar is said, the word 'judgment' is used in the same sense as I have connected with the word 'thought,' so that I would use the formulation: 'A hypothetical thought establishes a reciprocal relationship between two thoughts.' This could be true only if an indefinite indicator is absent;* but in such a case there would also be no generality.

If an instant of time is to be indefinitely indicated in both conditional and dependent clauses, this is often achieved merely by using the present tense of the verb, which in such a case however does not indicate the temporal present. This grammatical form is then the indefinite indicator in the main and subordinate clauses. An example of this is: 'When the Sun is in the tropic of Cancer, the longest day in the northern hemisphere

from Denmark, Prussia and Austria quarrelled.' In this version, it is surely sufficiently clear that the sense is not to be taken as having as a part the thought that Schleswig-Holstein was once separated from Denmark, but that this is the necessary presupposition in order for the expression 'after the separation of Schleswig-Holstein from Denmark' to have any reference at all. To be sure, our sentence can also be interpreted as saying that Schleswig-Holstein was once separated from Denmark. We then have a case which is to be considered later. In order to understand the difference more clearly, let us project ourselves into the mind of a Chinese who, having little knowledge of European history, believes it to be false that Schleswig-Holstein was ever separated from Denmark. He will take our sentence, in the first version, to be neither true nor false but will deny it to have any reference, on the ground of absence of reference for its subordinate clause. This clause would only apparently determine a time. If he interpreted our sentence in the second way, however, he would find a thought expressed in it which he would take to be false, beside a part which would be without reference for him.

* At times an explicit linguistic indication is missing and must be read off from the entire context.

occurs.' Here, also, it is impossible to express the sense of the subordinate clause in a full sentence, because this sense is not a complete thought. If we say: 'The Sun is in the tropic of Cancer,' this would refer to our present time and thereby change the sense. Just as little is the sense of the main clause a thought; only the whole, composed of main and subordinate clauses, has such a sense. It may be added that several common components in the antecedent and consequent clauses may be indefinitely indicated.

It is clear that noun clauses with 'who' or 'what' and adverbial clauses with 'where,' 'when,' 'wherever,' 'whenever' are often to be interpreted as having the sense of conditional clauses, e.g. 'who touches pitch, defiles himself.'

Adjective clauses can also take the place of conditional clauses. Thus the sense of the sentence previously used can be given in the form 'The square of a number which is less than 1 and greater than 0 is less than 1 and greater than 0.'

The situation is quite different if the common component of the two clauses is designated by a proper name. In the sentence:

Napoleon, who recognized the danger to his right flank, himself led his guards against the enemy position

two thoughts are expressed:

- 1 Napoleon recognized the danger to his right flank
- 2 Napoleon himself led his guards against the enemy position.

When and where this happened is to be fixed only by the context, but is nevertheless to be taken as definitely determined thereby. If the entire sentence is uttered as an assertion, we thereby simultaneously assert both component sentences. If one of the parts is false, the whole is false. Here we have the case that the subordinate clause by itself has a complete thought as sense (if we complete it by indication of place and time). The reference of the subordinate clause is accordingly a truth value. We can therefore expect that it may be replaced, without harm to the truth value of the whole, by a sentence having the same truth value. This is indeed the case; but it is to be noticed that for purely grammatical reasons, its subject must be 'Napoleon,' for only then can it be brought into the form of an adjective clause belonging to 'Napoleon.' But if the demand that it be expressed in this form be waived, and the connexion be shown by 'and,' this restriction disappears.

Subsidiary clauses beginning with 'although' also express complete thoughts. This conjunction actually has no sense and does not change the sense of the clause but only illuminates it in a peculiar fashion.* We could indeed replace the concessive clause without harm to the truth of the whole by another of the same truth value; but the light in which the clause is placed by the conjunction might then easily appear unsuitable, as if a song with a sad subject were to be sung in a lively fashion.

In the last cases the truth of the whole included the truth of the component clauses. The case is different if a conditional clause expresses a complete thought by containing, in place of an indefinite indicator, a proper name or something which is to be regarded as equivalent. In the sentence

If the Sun has already risen, the sky is very cloudy

the time is the present, that is to say, definite. And the place is also to be thought of as definite. Here it can be said that a relation between the truth values of conditional and dependent clauses has been asserted, viz. such that the case does not occur in which the antecedent stands for the True and the consequent for the False. Accordingly, our sentence is true if the Sun has not yet risen, whether the sky is very cloudy or not, and also if the Sun has risen and the sky is very cloudy. Since only truth values are here in question, each component clause can be replaced by another of the same truth value without changing the truth value of the whole. To be sure, the light in which the subject then appears would usually be unsuitable; the thought might easily seem distorted; but this has nothing to do with its truth value. One must always take care not to clash with the subsidiary thoughts, which are however not explicitly expressed and therefore should not be reckoned in the sense. Hence, also, no account need be taken of their truth values.†

The simple cases have now been discussed. Let us review what we have learned.

The subordinate clause usually has for its sense not a thought, but only a part of one, and consequently no truth value as reference. The reason for this is either that the words in the subordinate clause have indirect reference, so that the reference, not the sense, of the subordinate clause is

* Similarly in the case of 'but,' 'yet.'

† The thought of our sentence might also be expressed thus: 'Either the Sun has not risen yet or the sky is very cloudy' – which shows how this kind of sentence connexion is to be understood.

a thought; or else that, on account of the presence of an indefinite indicator, the subordinate clause is incomplete and expresses a thought only when combined with the main clause. It may happen, however, that the sense of the subsidiary clause is a complete thought, in which case it can be replaced by another of the same truth value without harm to the truth of the whole – provided there are no grammatical obstacles.

An examination of all the subordinate clauses which one may encounter will soon provide some which do not fit well into these categories. The reason, so far as I can see, is that these subordinate clauses have no such simple sense. Almost always, it seems, we connect with the main thoughts expressed by us subsidiary thoughts which, although not expressed, are associated with our words, in accordance with psychological laws, by the hearer. And since the subsidiary thought appears to be connected with our words of its own accord, almost like the main thought itself, we want it also to be expressed. The sense of the sentence is thereby enriched, and it may well happen that we have more simple thoughts than clauses. In many cases the sentence must be understood in this way, in others it may be doubtful whether the subsidiary thought belongs to the sense of the sentence or only accompanies it.* One might perhaps find that the sentence

Napoleon, who recognized the danger to his right flank, himself led his guards against the enemy position

expresses not only the two thoughts shown above, but also the thought that the knowledge of the danger was the reason why he led the guards against the enemy position. One may in fact doubt whether this thought is merely slightly suggested or really expressed. Let the question be considered whether our sentence be false if Napoleon's decision had already been made before he recognized the danger. If our sentence could be true in spite of this, the subsidiary thought should not be understood as part of the sense. One would probably decide in favour of this. The alternative would make for a quite complicated situation: We would have more simple thoughts than clauses. If the sentence

Napoleon recognized the danger to his right flank

were now to be replaced by another having the same truth value, e.g.

Napoleon was already more than 45 years old

* This may be important for the question whether an assertion is a lie, or an oath a perjury.

not only would our first thought be changed, but also our third one. Hence the truth value of the latter might change – viz. if his age was not the reason for the decision to lead the guards against the enemy. This shows why clauses of equal truth value cannot always be substituted for one another in such cases. The clause expresses more through its connexion with another than it does in isolation.

Let us now consider cases where this regularly happens. In the sentence:

Bebel fancies that the return of Alsace-Lorraine would appease France's desire for revenge

two thoughts are expressed, which are not however shown by means of antecedent and consequent clauses, viz.:

- (1) Bebel believes that the return of Alsace-Lorraine would appease France's desire for revenge
- (2) the return of Alsace-Lorraine would not appease France's desire for revenge.

In the expression of the first thought, the words of the subordinate clause have their indirect reference, while the same words have their customary reference in the expression of the second thought. This shows that the subordinate clause in our original complex sentence is to be taken twice over, with different reference, standing once for a thought, once for a truth value. Since the truth value is not the whole reference of the subordinate clause, we cannot simply replace the latter by another of equal truth value. Similar considerations apply to expressions such as 'know,' 'discover,' 'it is known that.'

By means of a subordinate causal clause and the associated main clause we express several thoughts, which however do not correspond separately to the original clauses. In the sentence: 'Because ice is less dense than water, it floats on water' we have

- (1) Ice is less dense than water;
- (2) If anything is less dense than water, it floats on water;
- (3) Ice floats on water.

The third thought, however, need not be explicitly introduced, since it is contained in the remaining two. On the other hand, neither the first and third nor the second and third combined would furnish the sense of our sentence. It can now be seen that our subordinate clause

because ice is less dense than water

expresses our first thought, as well as a part of our second. This is how it comes to pass that our subsidiary clause cannot be simply replaced by another of equal truth value; for this would alter our second thought and thereby might well alter its truth value.

The situation is similar in the sentence

If iron were less dense than water, it would float on water.

Here we have the two thoughts that iron is not less dense than water, and that something floats on water if it is less dense than water. The subsidiary clause again expresses one thought and a part of the other.

If we interpret the sentence already considered

After Schleswig-Holstein was separated from Denmark, Prussia and Austria quarrelled

in such a way that it expresses the thought that Schleswig-Holstein was once separated from Denmark, we have first this thought, and secondly the thought that at a time, more closely determined by the subordinate clause, Prussia and Austria quarrelled. Here also the subordinate clause expresses not only one thought but also a part of another. Therefore it may not in general be replaced by another of the same truth value.

It is hard to exhaust all the possibilities given by language; but I hope to have brought to light at least the essential reasons why a subordinate clause may not always be replaced by another of equal truth value without harm to the truth of the whole sentence structure. These reasons arise:

- (1) when the subordinate clause does not stand for a truth value, inasmuch as it expresses only a part of a thought;
- (2) when the subordinate clause does stand for a truth value but is not restricted to so doing, inasmuch as its sense includes one thought and part of another.

The first case arises:

- (a) in indirect reference of words
- (b) if a part of the sentence is only an indefinite indicator instead of a proper name.

In the second case, the subsidiary clause may have to be taken twice over, viz. once in its customary reference, and the other time in indirect reference; or the sense of a part of the subordinate clause may likewise be a component of another thought, which, taken together with the thought directly expressed by the subordinate clause, makes up the sense of the whole sentence.

It follows with sufficient probability from the foregoing that the cases where a subordinate clause is not replaceable by another of the same value cannot be brought in disproof of our view that a truth value is the reference of a sentence having a thought as its sense.

Let us return to our starting point.

When we found ' $a = a$ ' and ' $a = b$ ' to have different cognitive values, the explanation is that for the purpose of knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its reference, i.e. its truth value. If now $a = b$, then indeed the reference of ' b ' is the same as that of ' a ,' and hence the truth value of ' $a = b$ ' is the same as that of ' $a = a$.' In spite of this, the sense of ' b ' may differ from that of ' a ,' and thereby the thought expressed in ' $a = b$ ' differs from that of ' $a = a$.' In that case the two sentences do not have the same cognitive value. If we understand by 'judgment' the advance from the thought to its truth value, as in the above paper, we can also say that the judgments are different.

Translator's Notes

- 1 The reference is to Frege's *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle, 1879).
- 2 See his 'Über Begriff und Gegenstand' (*Vierteljahrsschrift für wissenschaftliche Philosophie* XVI [1892], 192–205).
- 3 See his 'Über Begriff und Gegenstand' (*Vierteljahrsschrift für wissenschaftliche Philosophie* XVI [1892], 192–205).
- 4 A literal translation of Frege's 'abstracten Nennsätzen' whose meaning eludes me.
- 5 Frege uses the Prussian name for the battle – 'Belle Alliance.'